

# Stability of Closed Timelike Geodesics in different Spacetimes.

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The linear stability of closed timelike geodesics (CTGs) is analyzed in two spacetimes with cylindrical sources, an infinite rotating dust cylinder, and a cylindrical cloud of static cosmic strings with a central spinning string. We also study the existence and linear stability of closed timelike curves in spacetimes that share some common features with the Gödel universe (Gödel-type spacetimes). In this case the existence of CTGs depends on the ‘background’ metric. The CTGs in a subclass of inhomogeneous stationary cosmological solutions of the Einstein-Maxwell equations with topology  $S^3 \times \mathbb{R}$  are also examined.

## I. INTRODUCTION

The existence of closed timelike curves (CTCs) presents a clear violation of causality. In some cases these CTCs can be disregarded because to have them one ought to have an external force acting along the whole CTC, process that will consume a great amount of energy. The energy needed to travel a CTC in Gödel universe is computed in [1]. For geodesics this is not the case since the external force is null, therefore the considerations of energy does not apply in this case and we have a bigger problem of breakdown of causality.

To the best of our knowledge we have four solutions to the Einstein equations that contain CTGs. One of them was given by Bonnor and Steadman [2] that studied the existence of CTGs in a spacetime with two spinning particles each one with magnetic moment equal to angular momentum and mass equal to charge (Perjeons), in particular, they present a explicit CTG. We found that this particular CTG is not stable, but there exist many other that are stable [3]. Soares [4] found a class of cosmological models, solutions of Einstein-Maxwell equations, with a subclass where the timelike paths of the matter are closed. For these models the existence of CTGs is demonstrated and explicit examples are given. Steadman [5] described the existence of CTGs in a vacuum exterior of the van Stockum solution for an infinite rotating dust cylinder. For this solution explicit examples of CTCs and CTGs are shown. And in [6] it is found CTGs in a spacetime associated to a cylindrical cloud of static strings with negative mass density with a central spinning string.

The possibility that a spacetime associated to a realistic model of matter may contain CTCs and, in particular, CTGs leads us to ask how permanent is the existence of these curves. Perhaps, one may rule out the CTCs by simple considerations about their linear stability. Otherwise, if these curves are stable under linear perturbations the conceptual problem associated to their existence is enhanced. Even though the matter content of the solutions listed before are far from realistic we shall consider the study the stability of these curves in order to see the possibility to rule them out only by dynamical considerations. These considerations lead us to study the stability of CTCs in Gödel universe [9].

In the present work, besides the study of the stability under linear perturbation of CTGs in the spacetimes described above, we shall also consider the existence and stability under linear perturbation of CTCs in the two examples of Gödel-type metrics given in [7], see also [8]. One of them has only CTCs and the other has CTGs depending on the choice of the parameters. All the cases analyzed are stationary and have axial symmetry.

It is interesting to note that these spacetimes are not counter examples of the Chronology Protection Conjecture [10] that essentially says that the laws of the physics do not allow the appearance of closed timelike curves. The spacetimes that we shall considered are given stationary spacetimes. A valid dynamic to built them is not known.

In Section 2 we present the general equations that will used to study linear stability. In Sections 3 and 4 we analyze the cases of a dust cylinder and a cylinder of cosmic strings with a central spinning string, respectively. In Section 5 we study two cases of Gödel-type metrics, one with flat background and the other with a conformally flat background. In Section 6 we consider the two explicit examples of CTGs given in [4]. And finally, in Section 7, we discuss and summarize ours results.

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## II. STABILITY OF CTCS AND CTGS

As we mentioned before the stability of CTCs, for the Gödel cosmological model are studied in [9]. Stability of geodesics are studied in [11] for particles moving around a black hole. Also in [12] and [13] considered the stability of geodesic moving on accretion disks and other structures.

Excepting the Soares CTGs all the others closed timelike geodesics that we shall study belong to spacetimes with metrics,  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ , where  $x^\mu = [t, r, \varphi, z]$ . In this case all the curves have the same parametric form,

$$t = t_*, \quad r = r_*, \quad \varphi \in [0, 2\pi], \quad z = z_*, \quad (1)$$

where  $t_*$ ,  $z_*$  and  $r_*$  are constants. The condition for these curves to be timelike is  $\frac{dx^\mu}{d\varphi} \frac{dx_\mu}{d\varphi} > 0$ , in other words,  $g_{\varphi\varphi} > 0$ .

A generic CTC  $\gamma$  satisfies the system of equations,

$$\ddot{x}^\mu + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \dot{x}^\beta = F^\mu(x), \quad (2)$$

where the overdot indicates derivation with respect to  $s$ ,  $\Gamma_{\alpha\beta}^\mu$  are the Christoffel symbols and  $F^\mu$  is a specific external force ( $a^\mu = F^\mu$ ). We have a closed timelike geodesic when  $a^\mu = 0$ ,  $\mu = 0, 1, 2, 3$ .

To analyze the linear stability of a CTG  $\gamma$  we consider a small perturbation  $\xi$ . The perturbed curve,  $\tilde{\gamma}$ , has the form  $\tilde{x}^\mu = x^\mu + \xi^\mu$ . From equations (2) one finds [9] that the system of differential equations satisfied by the perturbation  $\xi$  is,

$$\frac{d^2 \xi^\alpha}{ds^2} + 2\Gamma_{\beta\mu}^\alpha \dot{x}^\mu \frac{d\xi^\beta}{ds} + \Gamma_{\beta\mu,\lambda}^\alpha \dot{x}^\mu \dot{x}^\lambda \xi^\lambda = F_{,\lambda}^\alpha \xi^\lambda, \quad (3)$$

where  $f_{,\lambda} \equiv \partial f / \partial x^\lambda$ . The coefficients of the metrics studied do not depend on  $\varphi$ . Therefore the coefficients of the linear system (3) are all constants and the analysis of stability in this case reduces to the solution of a linear system of equations with constant coefficients. Also in all cases we will have conservation of the angular momentum that for the studied CTGs will imply that the angular velocity  $\dot{\varphi}$  will be a positive constant.

Excepting Soares CTGs, all the other are circles in the  $(r, \varphi)$ -plane. We will have a CTG when the function  $s(r, z_*) = 2\pi\sqrt{g_{\varphi\varphi}(r, z_*)}$  presents a local maximum at some value of  $r$ , say  $r = \bar{r}$ . It is important to stress that to have a reasonable CTG it is necessary that  $\bar{r} \neq 0$  and  $g_{\varphi\varphi}(\bar{r}, z_*) > 0$ . Moreover, a way to obtain CTGs in a spacetime that only has CTCs is to deform this last spacetime, for example, adding matter in such way that the function  $g_{\varphi\varphi}(r, z_*)$  of the deformed spacetime be a function with a local maximum at  $r = \bar{r}$  with  $\bar{r} \neq 0$  and  $g_{\varphi\varphi}(\bar{r}, z_*) > 0$ .

## III. VAN STOCKUM SOLUTION

Steadman [5] described the behavior of CTGs in the exterior of the van Stockum solution for an infinite rotating dust cylinder. The metric is expressed in Weyl-Papapetrou coordinates as,

$$ds^2 = Fdt^2 - H(dr^2 + dz^2) - Ld\varphi^2 - 2Md\varphi dt. \quad (4)$$

The metric coefficients in the interior of the cylinder are,

$$H = e^{-a^2 r^2}, \quad L = r^2(1 - a^2 r^2), \quad \rho = 4a^2 e^{a^2 r^2}, \quad M = ar^2, \quad F = 1, \quad (5)$$

where  $a$  is the angular velocity of the cylinder and  $\rho$  the matter density.

In order to have no superluminal matter it is required that radius of the cylinder be less than  $1/a$ , i.e., at the boundary  $r = R < 1/a$ . For the closed curve  $\gamma$  given in (1) this condition does not allow CTCs inside the cylinder.

Van Stockum found a procedure which generates an exterior solution for all  $aR > 0$ . He divided this solution in three possibilities, depends on the value of  $aR$ . We have CTCs when  $aR > 1/2$ , in this case, the exterior solution is

$$\begin{aligned} H &= e^{-a^2 R^2} (r/R)^{-2a^2 R^2}, \\ L &= \frac{Rr \sin(3\beta + \ln(r/R) \tan \beta)}{2 \sin 2\beta \cos \beta}, \\ M &= \frac{r \sin(\beta + \ln(r/R) \tan \beta)}{\sin 2\beta}, \\ F &= \frac{r \sin(\beta - \ln(r/R) \tan \beta)}{R \sin \beta}, \end{aligned} \quad (6)$$

with

$$\tan \beta = \sqrt{4a^2 R^2 - 1}, \quad \frac{1}{2} < aR < 1. \quad (7)$$

In the definition of  $\tan \beta$ , we take the positive square root and the principal value of  $\beta$ . With these restrictions, it is possible [5] to find closed timelike geodesics in this exterior solution.

For the exterior metric (6), the curve  $\gamma$  is timelike when  $g_{\varphi\varphi} = -L > 0$  and this occurs when  $r_*$  belongs to open interval  $R_k$ , where

$$R_k = \left( R \exp \left[ \frac{(2k-1)\pi - 3\beta}{\tan \beta} \right], R \exp \left[ \frac{2k\pi - 3\beta}{\tan \beta} \right] \right), \quad k \in \mathcal{Z}. \quad (8)$$

The four-acceleration of  $\gamma$  has only one non zero component,

$$a^r = \frac{e^{a^2 R^2} (r_*/R)^{2a^2 R^2} \sin(4\beta + \ln(r_*/R) \tan \beta)}{2r_* \cos \beta \sin(3\beta + \ln(r_*/R) \tan \beta)}. \quad (9)$$

The radial coordinates of geodesics are the solutions of  $a^r(r_*) = 0$ . There are an infinite number of solutions and those occurring in the regions  $R_k$  are,

$$r_* = r_k = R e^{2(k\pi - 2\beta) \cot \beta}, \quad k = 1, 2, 3, \dots \quad (10)$$

The radial coordinates of the CTGs coincide with the local maximum of  $g_{\varphi\varphi}$ .

For the above mentioned closed timelike geodesics the system (3) reduces to

$$\begin{aligned} \ddot{\xi}^0 + k_1 \dot{\xi}^1 &= 0, \\ \ddot{\xi}^1 + k_2 \dot{\xi}^0 + k_3 \dot{\xi}^1 &= 0, \\ \ddot{\xi}^2 + k_4 \dot{\xi}^1 &= 0, \\ \ddot{\xi}^3 &= 0, \end{aligned} \quad (11)$$

where

$$k_1 = 2\Gamma_{21}^0 \dot{\varphi}, \quad k_2 = 2\Gamma_{20}^1 \dot{\varphi}, \quad k_3 = \Gamma_{22,1}^1 \dot{\varphi}^2, \quad k_4 = 2\Gamma_{21}^2 \dot{\varphi}. \quad (12)$$

The solution of (11) is

$$\begin{aligned} \xi^0 &= -k_1 (c_3 \sin(\omega s + c_4)/\omega + \lambda s) + c_1 s + c_5, \\ \xi^1 &= c_3 \cos(\omega s + c_4) + \lambda, \\ \xi^2 &= -k_4 (c_3 \sin(\omega s + c_4)/\omega + \lambda s) + c_2 s + c_6, \\ \xi^3 &= c_7 s + c_8, \end{aligned} \quad (13)$$

where  $c_i$ ,  $i = 1, \dots, 8$  are integration constants,

$$\omega = \sqrt{k_3 - k_1 k_2} \quad (14)$$

$$= \left[ \left( \frac{r_k}{R} \right)^{2a^2 R^2 - 1} \frac{e^{a^2 R^2}}{8 \cos^4 \beta} \dot{\varphi}^2 \right]^{1/2}, \quad (15)$$

and  $\lambda = -k_2 c_1 / \omega^2$ .

Since  $\omega > 0$  is real the solution (13) shows the typical behavior for stability, we have vibrational modes untangled with translational ones that can be eliminated by a suitable choice of the initial conditions.

#### IV. CLOUD OF COSMIC STRINGS

Now we shall consider the spacetime associated to an infinite cylinder formed by a cloud of parallel static strings with a central spinning cosmic string [6]. The clouds of cosmic strings were introduced in [14], see also [15], and for spinning strings see [16, 17]. The spacetime related to the interior of the cylinder of finite radius,  $r = R$ , is matched continuously to the external metric that is taken as representing a rotating cosmic string.

The line element inside cylinder is

$$ds^2 = (dt - k d\varphi)^2 - [D^2(r)d\varphi^2 + dr^2 + dz^2]. \quad (16)$$

We have closed timelike curves when  $k^2 - D(r_*) > 0$ . The nonzero component of the four-acceleration of this curve is given by  $a^r = D'(r)$ . Therefore, when  $D'(r_*) = 0$  the curve  $\gamma$  is a geodesic.

The nonzero contravariant components of the energy-momentum tensor in the cloud are  $T^{tt} = \rho$  and  $T^{zz} = p = -\rho$ . The Einstein equations in this case reduce to the single equation,

$$\frac{D''}{D} = -\rho. \quad (17)$$

The sign analysis of  $D''(r)$  and  $D(r)$  shows that if there exist CTGs and the condition  $D''(r)D(r) < 0$  holds, then  $D(r)$  changes sign at least once. Therefore, there are values of  $r = \bar{r}$  where  $D(\bar{r}) = 0$ , i.e., the metric is degenerate. In order to obtain a connected spacetime it is assumed that the mass density of the cloud is negative. Hence  $D''(r)D(r) > 0$  for  $r \leq R$ .

For the metric (16), the system (3) is

$$\begin{aligned} \ddot{\xi}^0 &= 0, \\ \ddot{\xi}^1 - D''(r_*)D(r_*)\dot{\varphi}^2\xi^1 &= 0 \\ \ddot{\xi}^2 &= 0, \quad \ddot{\xi}^3 = 0. \end{aligned} \quad (18)$$

We have a solution with periodic modes when  $D''(r_*)D(r_*) < 0$ . But it was assumed that  $D''(r)D(r) > 0$  for  $r \leq R$ , then the CTGs inside the cylinder are not stable.

The line element outside the cylinder is assumed to be the one of a spinning string,

$$ds^2 = (dt - 8\pi J d\varphi)^2 - (1 - 8\pi\lambda)^2 r^2 d\varphi^2 - (dr^2 + dz^2), \quad (19)$$

where  $\lambda$  is the string linear density and  $J$  the spin angular momentum per length unit. There exist CTCs outside cloud when  $r_* < 8\pi|J|/|1 - 8\pi\lambda|$ , but, we do not have CTGs. This can be proved by analyzing the existence of CTCs with maximum length outside of the cylinder. The function that gives the length of these CTCs is  $s(r) = 2\pi[(8\pi J)^2 - (1 - 8\pi\lambda)^2 r^2]^{1/2}$  that has maximum point only at  $r = 0$ . Moreover, the length of CTCs inside the cylinder is given by  $s(r) = 2\pi\sqrt{k^2 - D^2(r)}$  and in order to have a CTG we need a function  $D(r)$  with a local maximum at  $\bar{r} \neq 0$  such that  $k^2 - D^2(\bar{r}) > 0$ . This occurs for the two examples given in [6]. They are (i)  $D(r) = \beta \cosh[(r - R)/r_0 + \alpha]$  and (ii)  $D(r) = c[(r - a)^2 + b^2]$ , where  $\alpha, \beta, a, b, c$ , and  $r_0$  are constants.

## V. THE GÖDEL-TYPE CASES

A Gödel-type (GT) metric  $g_{\mu\nu}$ , as defined in [7], is a  $D$ -dimensional metric of the form

$$g_{\mu\nu} = u_\mu u_\nu - h_{\mu\nu}, \quad (20)$$

where the ‘background’  $h_{\mu\nu}$  is the metric of a  $(D - 1)$ -dimensional spacetime perpendicular to the timelike unit vector  $u^\mu$ . Further more we assume that  $h_{\mu\nu}$  and  $u_\mu$  are independent of the fixed special coordinate  $x^k$  with  $0 \leq k \leq D - 1$  and, moreover, that  $h_{k\mu} = 0$ .

We shall consider the special cases with  $D = 4$  (four dimensional spacetime) and constant  $u_k$ . Also we assume, without losing generality, that the special fixed coordinate  $x^k$  is  $x^0 \equiv t$ , then  $h_{0\mu} = 0$ . We also do  $u_0 = 1$ .

The Gödel-type metric (20) solves the Einstein-Maxwell dust field equations in four dimensions provides the flat three-dimensional Euclidean source-free Maxwell equations

$$\partial_i f_{ij} = 0, \quad (21)$$

holds, where  $f_{\alpha\beta} \equiv \partial_\alpha u_\beta - \partial_\beta u_\alpha$ .

### A. GT-metrics with flat background

First, let us consider a Gödel-type metric with flat background. In the usual cylindrical coordinates  $(r, \varphi, z)$  the line element for this spacetime is,

$$ds^2 = (dt - \alpha r^2 d\varphi)^2 - (dr^2 + r^2 d\varphi^2 + dz^2). \quad (22)$$

The curve (1),  $\gamma$ , is timelike when  $g_{\varphi\varphi} = (\alpha^2 r_*^2 - 1)r_*^2 > 0$  that leads us to the condition,

$$r_*^2 > 1/\alpha^2. \quad (23)$$

For the CTC  $\gamma$  we find that the nonzero component of the four-acceleration satisfies  $a^r = r_*(2\alpha^2 r_*^2 - 1)\dot{\varphi}^2$ . The component  $a^r$  is identically null when  $r_*^2 = 1/2\alpha^2$ . Therefore the condition for  $\gamma$  to be timelike (23) is not satisfied and then this curve can not be a closed timelike geodesic.

For this CTC the system of perturbation (3) can be written as

$$\begin{aligned} \ddot{\xi}^0 + k_1 \dot{\xi}^1 &= 0, \\ \ddot{\xi}^1 + k_2 \dot{\xi}^0 + k_3 \dot{\xi}^2 + k_4 \xi^1 &= 0, \\ \ddot{\xi}^2 + k_5 \dot{\xi}^1 &= 0 \\ \ddot{\xi}^3 &= 0. \end{aligned} \quad (24)$$

where

$$k_1 = 2\Gamma_{12}^0 \dot{\varphi}, k_2 = 2\Gamma_{02}^1 \dot{\varphi}, k_3 = \Gamma_{22}^1 \dot{\varphi}^2, k_4 = -\Gamma_{22}^1 \partial_r(\dot{\varphi}^2), k_5 = 2\Gamma_{12}^2 \dot{\varphi}. \quad (25)$$

The solution of system (24) is given by:

$$\begin{aligned} \xi^0 &= -k_1(c_3 \sin(\omega s + c_4)/\omega + \lambda s) + c_1 s + c_5, \\ \xi^1 &= c_3 \cos(\omega s + c_4) + \lambda, \\ \xi^2 &= -k_5(c_3 \sin(\omega s + c_4)/\omega + \lambda s) + c_2 s + c_6, \\ \xi^3 &= c_7 s + c_8, \end{aligned} \quad (26)$$

where  $c_i$ ,  $i = 1, \dots, 8$  are integration constants,

$$\omega = \sqrt{k_4 - k_1 k_2 - k_3 k_4}, \quad (27)$$

$$= \left( \frac{2(2\alpha^6 r_*^6 - 4\alpha^4 r_*^4 + 4\alpha^2 r_*^2 - 1)\dot{\varphi}^2}{r_*^2(\alpha^2 r_*^2 - 1)^2} \right)^{1/2}, \quad (28)$$

and  $\lambda = -k_2 c_1 / \omega^2$ . Thus the CTC is linearly stable when  $\alpha r_* > 0.6$ .

## B. GT-metrics with conformally flat background

Now we shall studied a Gödel-type metric with a conformally flat background, in this case the line element is,

$$ds^2 = (dt - \frac{1}{\rho^4}(a + \rho^3 b)\alpha r^2 d\varphi)^2 - \frac{1}{\rho^4}(dr^2 + r^2 d\varphi^2 + dz^2). \quad (29)$$

where  $\rho$  is the radial distance in  $R^3$ ,  $\rho = \sqrt{r^2 + z^2}$ .

From the geodesic equations we find that the nonzero components of the four-acceleration associated to the curve (1) are,

$$\begin{aligned} a^r &= -\frac{r_*}{\rho_0^2}[\rho_0^2(1 - 2\alpha^2 r_*^2(a + b\rho_0^3)^2) + r_*^2(r_*^2 \alpha^2(\rho_0^3 ab - \rho_0^6 b^2 + 2a^2) - 2)]\dot{\varphi}^2, \\ a^z &= \frac{z_*}{\rho_0^2}r_*^2(\alpha^2 r_*^2(\rho_0^3 ab - \rho_0^6 b^2 + 2a^2) - 2)\dot{\varphi}^2 \end{aligned} \quad (30)$$

When  $z_* = 0$  we have  $a^z = 0$  and we are left with only one nonzero component of the acceleration,

$$a^r = \frac{1}{r_*}(\alpha^2(br_*^3 + a)(br_*^3 - 2a) + r_*^2)\dot{\varphi}^2. \quad (31)$$

We have also  $g_{\varphi\varphi} = (\alpha^2(a + br_*^3)^2 - r_*^2)/r_*^4$ . It is possible to choose parameters  $\alpha$ ,  $a$  and  $b$  such that  $a^r(r_*) = 0$  has positive roots (see for instance the values presented at the end of this Sub-Section). Therefore we have CTGs for these values of  $r_*$ . In this case, for these CTGs the system (3) reduces to

$$\begin{aligned} \ddot{\xi}^0 + k_1 \dot{\xi}^1 &= 0, \\ \ddot{\xi}^1 + k_2 \dot{\xi}^0 + k_3 \dot{\xi}^1 &= 0, \\ \ddot{\xi}^2 + k_4 \dot{\xi}^1 &= 0, \\ \ddot{\xi}^3 &= 0, \end{aligned} \quad (32)$$

where

$$k_1 = 2\Gamma_{21}^0 \dot{\varphi}, k_2 = 2\Gamma_{20}^1 \dot{\varphi}, k_3 = \Gamma_{22,1}^1 \dot{\varphi}^2, k_4 = 2\Gamma_{21}^2 \dot{\varphi}.$$

The solution of (32) is

$$\begin{aligned}\xi^0 &= -k_1(c_3 \sin(\omega s + c_4)/\omega + \lambda s) + c_1 s + c_5, \\ \xi^1 &= c_3 \cos(\omega s + c_4) + \lambda, \\ \xi^2 &= -k_4(c_3 \sin(\omega s + c_4)/\omega + \lambda s) + c_2 s + c_6, \\ \xi^3 &= c_7 s + c_8,\end{aligned}\tag{33}$$

where  $c_i$ ,  $i = 1, \dots, 8$  are integration constants and

$$\omega = \left[ \left( 1 + \frac{a + br_*^3}{r_*^4} (2\alpha^2 r_*^2 - \alpha^4 (br_*^3 - 2a^2)) \right) \dot{\varphi}^2 \right]^{1/2}\tag{34}$$

In order to have  $\omega^2 > 0$  it is necessary that the second term inside of branches be positive or less than one. To do this we can keep  $b$  small. For example: a) choosing  $a = b = \alpha = 1$ , we have  $r_* = 1.138684455$  and  $g_{\varphi\varphi} = 2.876589804$  and  $\omega^2 = 9.459585855\dot{\varphi}^2$ , b) for  $a = \alpha = 1$  and  $b = 0.5$ , we have  $r_* = 1.333767038$ ,  $g_{\varphi\varphi} = 0.9483504120$  and  $\omega^2 = 5.374106554\dot{\varphi}^2$ , and c) for  $a = 2$ ,  $\alpha = 1$  and  $b = 0.1$ , we have  $r_* = 2.775627312$ , and  $g_{\varphi\varphi} = 0.1587448306$  and  $\omega^2 = 4.44597413\dot{\varphi}^2$ . Therefore these three examples represent stable CTGs.

## VI. THE SOARES COSMOLOGICAL MODEL CASE.

This model describes a class of inhomogeneous stationary cosmological solutions of Einstein-Maxwell equations, with rotating dust and electromagnetic field [4]. We are interested in the subclass of these models with spacetime topology  $S^3 \times \mathbb{R}$  and with the dust moving along closed timelike geodesics.

For the metric

$$ds^2 = A_0^2(dt - 2\lambda_1 \cos \theta d\varphi)^2 - dr^2 - B_0^2(d\theta^2 + \sin^2 \theta d\varphi^2),\tag{35}$$

where  $B_0^2 = k\Sigma^2 - A_0^2\lambda_1^2$  and  $A_0$ ,  $\lambda_1$ ,  $\Sigma$  and  $k$  are constants. Note that by definition the time coordinate is a periodic variable [4].

The nonspacelike geodesics are described by the tangent vector field  $\dot{x}^\alpha = dx^\alpha/ds$ , with

$$\begin{aligned}\dot{t} &= k_0 + \lambda_1 \cos \theta \frac{h_0 + k_0 A_0^2 \lambda_1 \cos \theta}{3A_0^2 \lambda_1^2 \cos^2 \theta - B_0^2 \sin^2 \theta}, \\ \dot{r} &= r_0, \\ \dot{\theta} &= \frac{(h_0 + k_0 A_0^2 \lambda_1 \cos \theta)^2 + A_0^2 k_0^2 - 1 - r_0}{B_0}, \\ \dot{\varphi} &= \frac{h_0 + k_0 A_0^2 \lambda_1 \cos \theta}{3A_0^2 \lambda_1^2 \cos^2 \theta - B_0^2 \sin^2 \theta},\end{aligned}$$

where  $h_0$ ,  $k_0$  and  $r_0$  are arbitrary parameters. Two trivial cases are given by choosing  $\theta = \theta_0 = \text{constant}$ .

**Case1.** Choose  $r_0$ ,  $h_0$ ,  $k_0$  such that

$$\begin{aligned}A_0^2 k_0^2 &= 1, \\ r_0 &= 0, \\ h_0 + k_0 A_0^2 \lambda_1 \cos \theta_0 &= 0.\end{aligned}\tag{36}$$

In this case (3) can be cast as,

$$\begin{aligned}\ddot{\xi}^0 + a \dot{\xi}^2 &= 0, \\ \ddot{\xi}^1 &= 0, \\ \ddot{\xi}^2 + b \dot{\xi}^3 &= 0, \\ \ddot{\xi}^3 + c \dot{\xi}^2 &= 0,\end{aligned}$$

where  $a = -4\lambda_1^2 \cos \theta_0 A_0^2 k_0^2 / (B_0^2 \sin \theta_0)$ ,  $b = -2\lambda_1 A_0^2 k_0^2 / (B_0^2 \sin \theta_0)$  and  $c = -2\lambda_1 k_0^2 \sin \theta_0 A_0^2 / B_0^2$ .

The solution of this system is given by

$$\begin{aligned}\xi^0 &= -a(c_5 \exp(\omega s)/\omega - c_6 \exp(-\omega s)/\omega + bc_4 s/\omega^2) + c_1 s + c_7; \\ \xi^1 &= c_2 s + c_3; \\ \xi^2 &= c_5 \exp(\omega s) + c_6 \exp(-\omega s) + bc_4/\omega^2; \\ \xi^3 &= -c(c_5 \exp(\omega s)/\omega - c_6 \exp(-\omega s)/\omega + bc_4 s/\omega^2) + c_4 s + c_8.\end{aligned}\tag{37}$$

where  $c_i$ ,  $i = 1, \dots, 8$  are integration constants, and  $\omega = \sqrt{bc}$ . Therefore these CTGs are not stable.

**Case2.** Choose  $h_0, k_0$  such that

$$\begin{aligned}A_0^2 k_0^2 &= 1 + r_0^2, \\ h_0 + k_0 A_0^2 \lambda_1 \cos \theta_0 &= 0.\end{aligned}\tag{38}$$

The system of perturbation is the same as before and also the CTGs are not stable.

## VII. DISCUSSION

In summary, we analyzed the linear stability of closed timelike geodesics in four solutions of Einstein's field equations. It is possible to find CTGs in different spacetimes, they can be fulfilled by matter or not and in both cases the CTGs can be stable or not.

In the first case, CTGs outside of an infinite dust cylinder, we found stable CTGs. In this model the closed curves are circles in a  $(r, \varphi)$ -plane  $\Pi = \{t = t_*, z = z_*\}$  and all conclusions obtained are independent of the values of the  $t_*$  and  $z_*$ . In  $\Pi$  the CTGs appears in infinitely many regions (open flat rings) filled by CTCs and these regions are separated by other regions where the closed curves are spacelike.

In the case of the cloud of cosmic strings there exist CTGs inside the source but these are not linearly stable. There are CTCs in the exterior, but no CTG. This is the only case when the matter content of the solution is exotic, i.e., it does not obey the usual energy conditions. We have negative matter density.

Examples of “cosmological” solutions with CTCs are Gödel-type metric with flat background, as well as, conformally flat background. The first has stable CTCs but no CTGs and for the second it is possible to find values for the parameters to have a spacetime with stable CTGs.

In the “cosmological” model described by Soares we found two examples of not stable CTGs. In this is a case the existence of the CTGs depends upon the nontrivial topology of spacetime.

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